

ORIENTED AREA IS A PERFECT MORSE FUNCTION

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ABSTRACT. We show that an appropriate generalization of the *oriented area function* is a perfect Morse function on the space of three-dimensional configurations of an equilateral polygonal linkage with odd number of edges. Therefore cyclic equilateral polygons (which appear as Morse points) are interpreted as independent generators of the homology groups of the (decorated) configuration space.

1. INTRODUCTION

A Morse function on a smooth manifold is called perfect if the number of critical points equals the sum of Betti numbers. Not every manifold possesses a perfect Morse function. Homological spheres (that are not spheres) do not possess it; manifolds with torsions in homologies do not possess it, etc. On the other hand, the celebrated Millnor-Smale cancelation technique for critical points with neighbor indices (or, equivalently, cancelation of handles) provides a series of existence-type theorems [1, 5] which are the key tool in Smale's proof of generalized Poincaré conjecture [8].

In the paper we focus on one particular example of a perfect Morse function and discuss some related problems. Namely, we bound ourselves by the space of configuration of the equilateral polygonal linkage with odd number $n = 2k + 1$ of edges. As the ambient space it makes sense to take either \mathbb{R}^2 or \mathbb{R}^3 , which gives us the spaces $M_2(n)$ and $M_3(n)$. In bigger dimension the configuration space is not a manifold. The number n is chosen to be odd by the same reason: for even n , the configuration space of the equilateral polygonal linkage has singular points.

We are interested in finding a "natural" perfect Morse function, that is, a function that has a transparent physical or geometrical meaning. The first candidate of a "natural" Morse function on $M_2(n)$ was the oriented area function A . Indeed, it is a Morse function with an easy description of its critical points as cyclic polygons (that is, polygons with a superscribed circle), and a simple formula for the Morse index of a critical point [4]. However, for $M_2(n)$ A is not perfect. In particular, for the equilateral pentagonal linkage it has one extra local maximum (except for the global maximum) and one extra local minimum, see Example 1. For the equilateral heptagonal linkage the number

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of Morse points greatly exceeds the sum of Betti numbers of the configuration space.

To build up a perfect Morse function, we take the space $M_3(n)$ and *decorate* it. The decorated space $\widetilde{M}_3(n)$ is well adjusted for an appropriate generalization S of the area function A . Its critical points (loosely speaking) are again cyclic polygons. Surprisingly, the function S is a perfect Morse function. As a direct corollary we interpret cyclic equilateral polygons as independent generators of the homology groups of the configuration space $\widetilde{M}_3(n)$.

2. PRELIMINARIES AND NOTATION

For an odd $n = 2k + 1$ an equilateral *polygonal n -linkage* should be interpreted as a collection of rigid bars of lengths 1 joined consecutively by revolving joints in a chain.

A *configuration* of the polygonal n -linkage in the Euclidean space \mathbb{R}^d , $d = 2, 3$ is a sequence of points $R = (p_1, \dots, p_{n+1})$, $p_i \in \mathbb{R}^d$ with $1 = |p_i, p_{i+1}|$ and $1 = |p_n, p_1|$ modulo the action of orientation preserving isometries of the space \mathbb{R}^d . We also call P a *polygon*. A configuration carries a natural orientation which we indicate in figures by an arrow.

The set $M_d(n)$ of all configurations up to an orientation-preserving isometry of the ambient space is *the moduli space, or the configuration space of the polygonal linkage L* .

For $d = 2, 3$ the space $M_d(n)$ is a smooth manifold.

We explain below in this paragraph what is known about planar configurations and the signed area function as the Morse function on the configuration space.

Definition 1. The *signed area* of a polygon $P \in M_2(n)$ with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \dots + (x_ny_1 - x_1y_n).$$

Definition 2. A polygon P is *cyclic* if all its vertices p_i lie on a circle.

Cyclic polygons arise here as critical points of the signed area: a polygon P is a critical point of the signed area function A iff P is a cyclic configuration. Their Morse indices are known [4, 10].

Example 1. [7] The equilateral pentagonal linkage has 14 cyclic configurations indicated in Fig. 4.

(1). The convex regular pentagon and its mirror image are the global maximum and minimum of the signed area A . Their Morse indices are 2 and 0 respectively.

(2). The starlike configurations are a local maximum and a local minimum of A .

(3). There are ten configurations that have three consecutive edges aligned. Their Morse indices equal 1.

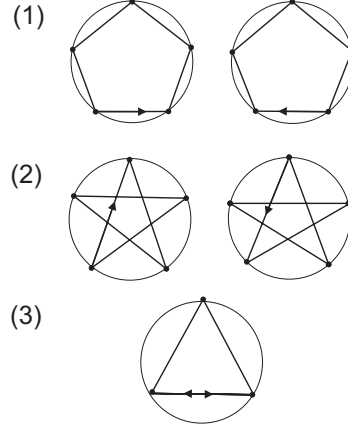


FIGURE 1. Cyclic configurations of the equilateral pentagonal linkage

3. THE DECORATED CONFIGURATION SPACE $\widetilde{\mathcal{M}}_3(n)$ AND THE AREA FUNCTION S

Definition 3. The *decorated configuration space* is defined as the set of pairs

$$\widetilde{M}_3(n) = \{(P, \xi) | P \text{ is a polygon in } \mathbb{R}^3 \text{ with the sidelengths 1; } \xi \in S^2, \}$$

factorized by the diagonal action of the orientation preserving isometries of \mathbb{R}^3 .

Here $S^2 \in \mathbb{R}^3$ is the unit sphere centered at the origin O .

Lemma 1. (1) *The space $\widetilde{M}_3(n)$ is an orientable fibration over $M_3(n)$ whose fiber is S^2 .*

(2) *The Euler class of the fibration equals zero.*

Proof. (1) The set of all polygons with fixed sidelengths (before factorization by isometries) is known to be orientable. Therefore the set of the pairs (a polygon, a vector) is also orientable as a trivial fibration. Since we take a factor by the action of orientation preserving isometries, the result is also orientable.

(2) $s(P) := \frac{\overrightarrow{p_1 p_2}}{|p_1 p_2|}$ defines a non-zero section. □

The Gysin sequence implies:

Corollary 1. (1) *We have the following short exact sequence:*

$$0 \rightarrow H^m(M(n)) \rightarrow H^m(\widetilde{M}(n)) \rightarrow H^{m-2}(M(n)) \rightarrow 0.$$

(2) *The homology groups $H_m(\widetilde{M}(n))$ are free abelian. For the Betti numbers we have:*

$$\beta^m(\widetilde{M}(n)) = \beta^m(M(n)) + \beta^{m-2}(M(n)). \quad \square$$

Definition 4. Let $(P, \xi) \in \widetilde{M}_3(n)$, let (x_i, y_i) be the vertices of P . The *area* of the pair (P, ξ) is defined as the scalar product:

$$2S(P, \xi) = (p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1, \xi).$$

An alternative equivalent definition is:

$$S(P, \xi) = A(pr_{\xi^\perp}(P)),$$

where pr_{ξ^\perp} is the plane orthogonal to ξ and cooriented by ξ .

Proposition 1. *For an equilateral polygon with odd number of edges, critical points (P, ξ) of the function S are pairs (P, ξ) such that P is a planar cyclic polygon, and ξ is orthogonal to the affine hull of P . If (P, ξ) is a critical point, then $(P, -\xi)$ is critical as well.*

Proof. The paper [6] contains a characterization of all critical points for a generic polygonal linkage (which is not necessarily equilateral). In our particular case it implies that critical points (P, ξ) of the function S fall into two classes:

- (1) **Planar cyclic configurations.** These are pairs (P, ξ) such that P is a planar cyclic polygon, and ξ is orthogonal to the affine hull of P .
- (2) **Non-planar configurations.** They are characterized by the three following conditions:
 - (a) The vectors ξ and $\vec{S} = p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1$ are parallel (but they can have opposite directions).
 - (b) The orthogonal projection of P onto the plane $\vec{S}(P)^\perp$ is a cyclic polygon.
 - (c) For every i , the vectors \vec{T}_i , \vec{S} , and \vec{d}_i are coplanar.

Here \vec{d}_i is the i -th short diagonal, \vec{T}_i is the vector area of the triangle $p_{i-1}p_i p_{i+1}$.

Let us show that the second class (non-planar configurations) is empty. Indeed, given a non-planar critical configuration, introduce a Cartesian system with the z -axis parallel to ξ . The three conditions (a), (b), and (c) imply that the absolute value of the slope of an edge with respect to the plane $(x, y) = \xi^\perp$ does not depend on the edge. This implies a contradiction with the closing condition: $\sum_{i=0}^n (z(p_i) - z(p_{i-1})) = 0$, where the indices are modulo n . \square

Theorem 1. (1) *The function S is a perfect Morse function on the decorated configuration space $\widetilde{M}_3(L)$ for an equilateral linkage with odd number of edges.*

- (2) *Each critical point of the function S is a pair (P, ξ) , where P is a planar cyclic configuration, ξ is a unit vector orthogonal to P . Each planar cyclic configuration P gives two critical points of the function S (with two different choices of the normal vector ξ).*
- (3) *The Morse index of a critical point (P, ξ) is*

$$m(P, \xi) = 2e - 2\omega - 2,$$

where ω is the winding number P around the center of the circumscribed circle, e is the number of edges that go counterclockwise.¹

Proof. (i) The second statement is already proven. Let us show that the number of critical points equals the sum of Betti numbers. The latter are already known due to A. Klyachko [3]:

$$\beta^{2p}(M_3(n)) = \sum_{0 \leq i \leq p} \binom{2k}{i}, \quad p < k.$$

By Corollary 1 we have

$$\beta^{2p}(\widetilde{M}_3(n)) = \sum_{0 \leq i \leq p} \binom{n}{i}, \quad p < k.$$

Each equilateral cyclic n -gon with an orthogonal vector ξ is defined by its winding number ω and by the set of edges that go clockwise. Assume that the winding number is positive (negative values are treated by symmetry). If the number of edges that go clockwise is e , then the winding number ranges from 1 to $(k - e)$.

For $p = 0, 1, \dots, k$ denote by N_n^p the number of such cyclic equilateral polygons for which $e - \omega - 1 = p$. Then $\widetilde{\beta}_n^{2p} = N_n^p$.

(iii) Straightforward analysis of the Hessian matrix is very complicated (probably impossible), so we use derive a combinatorial approach. We prove the formula for Morse index by induction by n . The base is given by equilateral pentagon. By symmetry, we assume in what follows that we have a critical point (P, ξ) such that $S(P, \xi) > 0$, or, equivalently, with winding number $\omega > 0$.

Proof is based on two observations:

(1) There is a natural embedding of a neighborhood of P in space $M_2(n)$ to $\widetilde{M}_3(n)$. It sends a configuration P to (P, ξ) with the same P and ξ orthogonal to the affine hull of P . The direction of ξ we choose in the way such that S is positive. Let us choose a basis of the tangent space $T_{(P, \xi)}\widetilde{M}_3(n)$ which starts by the pushforward of a basis of $T_P M_2(n)$ and ends by coordinates of ξ . The Hessian matrix related to this basis is a block matrix:

$$HESS(P, \xi) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & -E \end{pmatrix},$$

where H_1 is the $(n - 3) \times (n - 3)$ Hessian matrix of the planar polygon P related to the area function A and the space $M_2(n)$; E is the 2×2 unit matrix.

(2) For an $(n + 2)$ -gon and a number $1 \leq i \leq n$, consider the embedding $\varphi_i : \widetilde{M}_3(n) \rightarrow \widetilde{M}_3(n + 2)$ which keeps ξ and replaces the edge number i by




¹ The vector ξ sets an orientation on the plane of the polygon, so it makes sense to speak of "edges going clockwise" and "edges going counterclockwise".







a fold of three edges (see Fig. 1, (3) for an equilateral triangle with an edge by a three-fold). Each critical point $(P, \xi) \in \widetilde{M}_3(n)$ induces a critical point $\varphi_i(P, \xi) \in \widetilde{M}_3(n+2)$. Since the embedding has codimension two, and all Morse indices can only be even, we have either $m(\varphi_i(P, \xi)) = m(P, \xi)$, $m(\varphi_i(P, \xi)) = 2 + m(P, \xi)$, or $m(\varphi_i(P, \xi)) = 4 + m(P, \xi)$. More precisely, replacement of an edge replaced by a three-fold adds two extra columns to H_1 and two extra columns to H_2 . We know from [4] that the Morse index of P related to $M_2(n)$ and the oriented area A (that is, the number of negative eigenvalues of H_1) equals $e - 2\omega - 1$. It increases by one after replacement of an edge by a three-fold. This only leaves the case $m(\varphi_i(P, \xi)) = 2 + m(P, \xi)$.

These arguments allows to make an induction step $n \rightarrow n + 2$ and thus proves (iii) for all cyclic configurations with triple edges.

It remains to prove the formula for configurations without triple edges, that is, with all the edges going counterclockwise. We keep assuming that $S(P, \xi) > 0$, so there are exactly k such polygons: with $\omega = 1, 2, 3, \dots, k$. Their Morse indices should be $2n - 4, 2n - 6, 2n - 8$, etc. The only question is which configuration has one or the other index. We know that H_1 contributes $n - 3, n - 5, n - 7$, etc. to each of the Morse indices. The block H_2 contributes not more than $n - 3$, and $-E$ contributes exactly two. The statement follows. \square

As an illustration, we list below all cyclic equilateral pentagons and heptagons. The first column depicts a combinatorial type, the third one tells the number of configurations of this type, and the last one tells the Morse index.

	orientation	number	Morse index
	1	1	6
	-1	1	0
	1	5	4
	-1	5	2
	1	1	4
	-1	1	2

polygon	orientation	number	Morse index
	1	1	10
	-1	1	0
	1	7	8
	-1	7	2
	1	7	6
	-1	7	4
	1	1	8
	-1	1	2
	1	21	6
	-1	21	4
	1	1	6
	-1	1	4

(a) Critical equilateral pentagons

(b) Critical equilateral heptagons

CONCLUDING REMARKS

Decorated configuration space and the area S can be defined for a polygonal linkage that is not necessarily equilateral. However, generically, the function S is not a perfect Morse function.

In the paper we omit the discussion about non-degeneracy of critical points, since it appears to be somewhat technical. However the following arguments work: We may replace the equilateral linkage $(1, 1, \dots, 1)$ by its perturbation $(1 + \epsilon_1, 1 + \epsilon_2, \dots, 1 + \epsilon_n)$ has non-degenerate critical configurations that are close to above described equilateral ones.

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